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Development of a Model for Induction Heating

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1 Derivation

The development of this derivation was taken from References [1] to [6]. The development leads to a set of equations that can be used to solve for the amount of induction heating within a conductor.

The following equations are a version of Maxwell's equations. Details of the intial assumptions to reach this state can be found in References [2] and [5].

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{1}$$

$$\nabla \times B = \mu J \tag{2}$$

$$\nabla \cdot B = 0 \tag{3}$$

$$\nabla \cdot E = 0 \tag{4}$$

Where

J	=	free charge current density vector (amp/ m^2)
E	=	electric field intensity (V/m)
H	=	magnetic field intensity (amp/m)
B	=	Magnetic flux density (webers/ m^2)
ρ	=	free charge density (coulombs/ m^3)
D	=	electric flux density vector (coulombs/ m^2)
μ	=	permeability (vs/a-m)
σ	=	conductivity (mhos/m)

J is in defined in conductors to be

$$J = \sigma E \tag{5}$$

In a conductor, we can substitute this relationship into Euquation 2 to give us

$$\nabla \times B = \mu J = \mu(\sigma E) \tag{6}$$

Then taking the divergence of this relationship.

$$\nabla \cdot (\nabla \times B) = \nabla \cdot (\mu \sigma E) \tag{7}$$
$$0 = \mu \sigma (\nabla \cdot E) \tag{8}$$

which implies

$$\nabla \cdot E = 0 \tag{9}$$

Going one step farther with the assumption of constant σ, we have

$$\nabla \cdot J = \nabla \cdot (\sigma E) = \sigma(\nabla E) \tag{10}$$
$$\therefore \nabla \cdot J = 0 \tag{11}$$

Next, we introduce a vector potential A and a scalar potential V as defined here.

$$B \equiv \nabla \times A \tag{12}$$
$$E \equiv -\nabla V - \frac{\partial A}{\partial t} \tag{13}$$

Substituting in the new vector potential A B in Equation 2 and then by indentity we have

$$\nabla \times (\nabla \times A) = \mu J \tag{14}$$
$$\nabla(\nabla \cdot A) - \nabla^2 A = \mu J \tag{15}$$

Recalling that $\nabla \cdot E = 0$ and substituting in the new scalar potential for E gives

$$\nabla \cdot (-\nabla V - \frac{\partial A}{\partial t}) = 0 \tag{16}$$
$$-\nabla^2 V - \frac{\partial(\nabla \cdot A)}{\partial t}) = 0 \tag{17}$$
$$\nabla^2 V = -\frac{\partial(\nabla \cdot A)}{\partial t}) \tag{18}$$

Requiring the vector potential A to be conserved, $\nabla \cdot A = 0$, gives us from Equation 15 $\nabla^2 A = -\mu J$ and in Equation 18 we have $\nabla^2 V = 0$.

In the absence of free charges, it is appropriate to set $V = 0$, which allows us to neglect the last equation, $\nabla^2 V = 0$. This then gives us

$$E \equiv -\frac{\partial A}{\partial t} \tag{19}$$

Knowing that $\nabla^2 A = -\mu J$ and then solving this equatio for A would alallow the determination of B from

$$B \equiv \nabla \times A \tag{20}$$

E could also be determined from

$$E = -\frac{\partial A}{\partial t} \tag{21}$$

At present, we do not really need B because we are seeking the power dissipated in the crucible, which can be determined from

$$P = J \cdot E \tag{22}$$

Now, IF J were independet of time and recalling that

$$J = \sigma E \tag{23}$$

$$E = -\frac{\partial A}{\partial t} \tag{24}$$

$$J = -\sigma \frac{\partial A}{\partial t} \tag{25}$$

$$\nabla^2 A = -\mu J \tag{26}$$

would imply that $A \neq F(t)$. In addition, this would imply that $E = -\frac{\partial A}{\partial t} = 0$, which can not be true because if this was true there would be no induction heating. But since we have alternating current power, $E \neq 0$.

Taking the equations above for a conductor and rearranging them gives

$$J = \sigma E \tag{27}$$

$$-\frac{1}{\mu} \nabla^2 A = \sigma \frac{-\partial A}{\partial t} \tag{28}$$

$$\nabla^2 A = \frac{1}{\sigma \mu} \frac{\partial A}{\partial t} \tag{29}$$

In an insulator ($\sigma = 0$)

$$\nabla^2 A = 0 \tag{30}$$

In the coil, we assume that J is known (prescribed and time varying)

$$\nabla^2 A = -\mu J \tag{31}$$

Up to this point, the solution has been discussed in general cartesian coordinates. If we were to assume axisymmetric behavior of the fields, the equations can be simplified even further. The coordinate system would depend on r, ϕ, and z. Assuming that the relevant quantities do not vary with ϕ allows us to define A as

$$A = e_r A_r(r, z, t) + e_\phi A_\phi(r, z, t) + e_z A_z(r, z, t) \tag{32}$$

Now, going back to the coil equation (Equation 31) and recalling that A is a vector in cylindrical coordinates gives us

$$\frac{\partial}{\partial r}(\frac{1}{\partial r}\frac{\partial}{\partial r}(rA_r)) + \frac{1}{r^2}\frac{\partial^2 A_r}{\partial \phi^2} - \frac{2}{r^2}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial^2 A_r}{\partial z} = -\mu J_r \tag{33}$$

$$\frac{\partial}{\partial r}(\frac{1}{\partial r}\frac{\partial}{\partial r}(rA_\theta)) + \frac{1}{r^2}\frac{\partial^2 A_\theta}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial \phi} + \frac{\partial^2 A_\phi}{\partial z^2} = -\mu J_\phi \tag{34}$$

$$\frac{\partial}{\partial r}(r\frac{\partial A_z}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 A_z}{\partial \phi^2} + \frac{\partial^2 A_z}{\partial z^2} = -\mu J_\phi \tag{35}$$

Eliminating all of the derivatives with respect to ϕ leaves

$$\frac{\partial}{\partial r}(\frac{1}{\partial r}\frac{\partial}{\partial r}(rA_r)) + \frac{\partial^2 A_r}{\partial z} = -\mu J_r \tag{36}$$

$$\frac{\partial}{\partial r}(\frac{1}{\partial r}\frac{\partial}{\partial r}(rA_\phi)) + \frac{\partial^2 A_\phi}{\partial z^2} = -\mu J_\phi \tag{37}$$

$$\frac{\partial}{\partial r}(r\frac{\partial A_z}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 A_z}{\partial \phi^2} + \frac{\partial^2 A_z}{\partial z^2} = -\mu J_\phi \tag{38}$$

By neglecting the helicity (pitch) of the coils as well as 3D effects, along with appropriate boundary conditions gives $A_r = 0$ and $A_z = 0$. This reduces the previous set of equations to

$$\frac{\partial}{\partial r}(\frac{1}{\partial r}\frac{\partial}{\partial r}(rA_\phi)) + \frac{\partial^2 A_\phi}{\partial z^2} = -\mu J_\phi \tag{39}$$

Because we know that A_r and A_z are both 0 we can conclude that

$$E = -\frac{\partial A}{\partial t} = -\frac{\partial A_\phi}{\partial t} \tag{40}$$

Also

$$B \equiv \nabla \times A \tag{41}$$

$$B_r = \frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \tag{42}$$

$$B_\phi = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \tag{43}$$

$$B_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial \phi} \tag{44}$$

Once again eliminating all derivatives with respect to ϕ leaves us with

$$B_r = -\frac{\partial A_\phi}{\partial z} \tag{45}$$

$$B_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) \tag{46}$$

It can be shown that this implies that $\nabla \cdot B = 0$ by examining the definition of this relationship in conjuction with the relationships for B_r ande B_z that were just found.

$$\nabla \cdot B = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} \tag{47}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(-r\frac{\partial A_\phi}{\partial z}) + \frac{\partial}{\partial z}(\frac{1}{\partial r}\frac{\partial}{\partial r}(rA_\phi)) \tag{48}$$

$$= \frac{-1}{r}\frac{\partial}{\partial r}(r\frac{\partial A_\phi}{\partial z}) + \frac{1}{r}\frac{\partial}{\partial z}(\frac{\partial}{\partial r}(rA_\phi)) \tag{49}$$

$$= \frac{-1}{r}\frac{\partial}{\partial r}(r\frac{\partial A_\phi}{\partial z}) + \frac{1}{r}\frac{\partial}{\partial r}(\frac{\partial}{\partial z}(rA_\phi)) \tag{50}$$

$$= 0 \tag{51}$$

Recall that we were left with

$$\frac{\partial}{\partial r}(\frac{1}{\partial r}\frac{\partial}{\partial r}(rA_\phi)) + \frac{\partial^2 A_\phi}{\partial z^2} = -\mu J_\phi \tag{52}$$

but, this formulation is not ameanable to Galerkin weak formulation, so we let

$$\psi(r, z, t) \equiv rA_\phi(r, z, t) \tag{53}$$

which gives

$$\frac{\partial}{\partial r}(\frac{1}{\partial r}(\frac{\partial \psi}{\partial r}) + \frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial \psi}{\partial z}) = -\mu J_\phi \tag{54}$$

This remarkably looks like $\nabla \cdot (K\nabla\psi)$, but do not forget that we are in cylindrical coordinates. In cylindrical coordinates, $\nabla \cdot (K\nabla\psi)$, would look like

$$\nabla \cdot (K\nabla\psi) = \frac{1}{\partial r}\frac{\partial}{\partial r}(rK\frac{\partial \psi}{\partial r}) + \frac{\partial}{\partial z}(K\frac{\partial \psi}{\partial z}) \tag{55}$$

where, $K = \frac{1}{r}$

We will return to the "form" of the equation later as it relates to being solved within a computational fluid dynamics package.

Now, assuming that a solution for $\psi(r, z, t)$ could be obtained, we could post process the solution to obtain

$$A_\phi = \frac{\psi}{r} \tag{56}$$

$$E_\phi = -\frac{1}{r}\frac{\partial \psi}{\partial t} \tag{57}$$

$$B_r = -\frac{1}{r}\frac{\partial \psi}{\partial z} = -K\frac{\partial \psi}{\partial z} \tag{58}$$

$$B_z = \frac{1}{r}\frac{\partial \psi}{\partial r} = K\frac{\partial \psi}{\partial r} \tag{59}$$

$$\tag{60}$$

where, E_ϕ is the only item needed to solve for the power density in a crucible.

For the type of engineering problem to be solved here, we want to remove the time-dependce of the problem by seeking the quasi-steady (periodic or time-harmonic) solution associated with the periodic driving force - the alternating current in the coil.

$$J_\phi = J_o(r,z)\exp^{i\omega t} \tag{61}$$

In the conductor we have $J = \sigma E$ and recall that

$$\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial \psi}{\partial z}) = -\mu J_\phi \tag{62}$$
$$\nabla \cdot (K\nabla\psi) = -\mu J_\phi \tag{63}$$

and that

$$E_\phi = -\frac{1}{r}\frac{\partial \psi}{\partial t} \tag{64}$$

This leads to the following equation in the coil

$$\nabla \cdot (K\nabla\psi) = -\mu J_o \exp^{i\omega t} \tag{65}$$

In the conductors we have

$$\nabla \cdot (K\nabla\psi) = -\mu(\sigma E_\phi) = \mu\sigma(-\frac{1}{r}\frac{\partial \psi}{\partial t}) \tag{66}$$
$$\nabla \cdot (K\nabla\psi) = \frac{\mu\sigma}{r}\frac{\partial \psi}{\partial t} \tag{67}$$

And in the dielectrics

$$\nabla \cdot (K\nabla\psi) = 0 \tag{68}$$

But ψ is complex and we to revert to real arithmetic we let

$$J_\phi = J_o \cos\omega t \tag{69}$$
$$\psi(r,z,t) = S(r,z)\sin\omega t + C(r,z)\cos\omega t \tag{70}$$

This results in two real equations everywhere within the solution domain. For the coil we have

$$\frac{\partial}{\partial r}(\frac{1}{\partial r}(\frac{\partial C}{\partial r}) + \frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial C}{\partial z}) = -\mu J_o \tag{71}$$
$$\frac{\partial}{\partial r}(\frac{1}{\partial r}(\frac{\partial S}{\partial r}) + \frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial S}{\partial z}) = 0 \tag{72}$$

In the conductors this results in

$$\frac{\partial}{\partial r}(\frac{1}{\partial r}(\frac{\partial C}{\partial r}) + \frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial C}{\partial z}) = \frac{\mu\sigma\omega}{r}S \tag{73}$$
$$\frac{\partial}{\partial r}(\frac{1}{\partial r}(\frac{\partial S}{\partial r}) + \frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial S}{\partial z}) = -\frac{\mu\sigma\omega}{r}C \tag{74}$$

Everywhere else we end up with

$$\frac{\partial}{\partial r}(\frac{1}{\partial r}(\frac{\partial C}{\partial r}) + \frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial C}{\partial z}) = 0 \tag{75}$$
$$\frac{\partial}{\partial r}(\frac{1}{\partial r}(\frac{\partial S}{\partial r}) + \frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial S}{\partial z}) = 0 \tag{76}$$

Rewriting these equations in tensor notation for cartesian coordinates and a variable $K = \frac{1}{r}$ gives us (coil, conductors, and elsewhere)

$$\nabla \cdot (K\nabla C) = -\mu J_o \tag{77}$$
$$\nabla \cdot (K\nabla S) = 0 \tag{78}$$

$$\nabla \cdot (K\nabla C) = \frac{\mu\sigma\omega}{r}S \tag{79}$$
$$\nabla \cdot (K\nabla S) = -\frac{\mu\sigma\omega}{r}C \tag{80}$$

$$\nabla \cdot (K\nabla C) = 0 \tag{81}$$
$$\nabla \cdot (K\nabla S) = 0 \tag{82}$$

If we solve these equations with the appropriate boundary conditions, they will yield $C(r,z)$ and $S(r,z)$. Using S and C, we must now determine the power deposited in the conductive components. Recall that power is

$$P = J \cdot E \tag{83}$$

In addition, recall that

$$J = \sigma E \tag{84}$$
$$E_\phi = -\frac{1}{r}\frac{\partial \psi}{\partial t} \tag{85}$$
$$\psi(r,z,t) = S(r,z)\sin(\omega t) + C(r,z)\cos(\omega t) \tag{86}$$

Making substitutions

$$J = \sigma E \tag{87}$$

$$P = J \cdot E = J_\phi E_\phi = (\sigma E_\phi)E_\phi = \sigma E_\phi^2 \tag{88}$$

$$= \sigma(-\frac{1}{r}\frac{\partial \psi}{\partial t})^2 \tag{89}$$

$$= \frac{\sigma}{r^2}(\frac{\partial \psi}{\partial t})^2 \tag{90}$$

Determining the time derivative of ψ gives

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial t}(\ S(r,z)\sin(\omega t) + C(r,z)\cos(\omega t)) \tag{91}$$

$$= S\omega\cos(\omega t) - C\omega\sin(\omega t) \tag{92}$$

Substituting back into the power relationship.

$$P = \frac{\sigma}{r^2}(\ S\omega\cos(\omega t) - C\omega\sin(\omega t))^2 \tag{93}$$

$$= \frac{\sigma\omega^2}{r^2}(\ S\cos(\omega t) - C\sin(\omega t))^2 \tag{94}$$

To determine the power deposition rate, we need to integrate this over one period.

$$Q(r,z) \equiv \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} P(r,z,t)dt \tag{95}$$

$$= \frac{\omega}{2\pi}\frac{\sigma\omega^2}{r^2}\int_0^{\frac{2\pi}{\omega}}(\ S^2\cos^2(\omega t) - SC\cos(\omega t)\sin(\omega t) + C^2\sin^2(\omega t))dt \tag{96}$$

$$\begin{aligned} = \frac{\omega}{2\pi}\frac{\sigma\omega^2}{r^2}\Bigg(&(\frac{S^2}{2\omega}\sin(\omega t)\cos(\omega t) + \frac{t}{2})\Big|_0^{\frac{2\pi}{\omega}} - SC(\frac{1}{2\omega}\sin^2(\omega t))\Big|_0^{\frac{2\pi}{\omega}} \\ &+ C^2(-\frac{1}{2\omega}\cos(\omega t)\sin(\omega t) + \frac{t}{2})\Big|_0^{\frac{2\pi}{\omega}}\Bigg) \end{aligned} \tag{97}$$

Which reduces to

$$Q(r,z) = \frac{\sigma\omega^2}{2r^2}(S^2 + C^2) \tag{98}$$

The penetration depth, or “skin” depth of the heating is defined as

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}} \tag{99}$$

2 FIDAP Implementation

As shown above, there are two coupled equations that must be solved in order to determine the power deposition. The numerical solution of these equations is needed in order to apply a source term within the energy equations.

These equations have previously solved in FIDAP. That implementation used modified versions of the momentum and energy equations to provide a mechanism for the solution of two coupled equations. Currently, we want to solve for the induction heating field in addition to the flow field and the energy equation. In order to do this, a mechanism has to be defined within FIDAP to solve these equations.

It is possible to solve for the transport of 15 different species within FIDAP. The goal is to be able to use two of the species transport equations to determine C and S, which would allow the calculation of a source term within the energy equation.

The species equations within FIDAP, take the following form in vector notation.

$$\rho_o(\frac{\partial c_n}{\partial t} + \vec{u} \cdot \nabla c_n) = \rho_o \nabla \cdot (\alpha_n \nabla c_n) + q_{cn} + R_n \tag{100}$$

For a steady problem that does not consider velocity for the source term R_n we have

$$\nabla \cdot (\alpha_n \nabla c_n) = -\frac{q_{cn}}{\rho_o} \tag{101}$$

Recall the governing equations for the coil, conductors, and all other locations

$$\nabla \cdot (K \nabla C) = -\mu J_o \tag{102}$$

$$\nabla \cdot (K \nabla S) = 0 \tag{103}$$

$$\nabla \cdot (K \nabla C) = \frac{\mu \sigma \omega}{r} S \tag{104}$$

$$\nabla \cdot (K \nabla S) = -\frac{\mu \sigma \omega}{r} C \tag{105}$$

$$\nabla \cdot (K \nabla C) = 0 \tag{106}$$

$$\nabla \cdot (K \nabla S) = 0 \tag{107}$$

This results in a situation where the source term q_{cn} within FIDAP has to be set to equal the right-hand-side of the governing equations. The diffusivity α_n is set equal to K which is defined as $\frac{1}{r}$ - a variable.

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